

Three-dimensional Chern-Simons black holes

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Abstract

We construct black hole solutions to three-dimensional Einstein-Maxwell theory with both gravitational and electromagnetic Chern-Simons terms. These intrinsically rotating solutions are geodesically complete, and causally regular within a certain parameter range. Their mass, angular momentum and entropy are found to satisfy the first law of black hole thermodynamics. These Chern-Simons black holes admit a four-parameter local isometry algebra, which generically is $sl(2, R) \times R$, and may be generated from the corresponding vacua by local coordinate transformations.

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1 Introduction

Three-dimensional gravity admits a variety of black hole solutions. The first discovered, and most well-known, is the Bañados-Teitelboim-Zanelli (BTZ) black hole solution to three-dimensional gravity with a negative cosmological constant [1]. Black holes in a generalized three-dimensional dilaton gravity theory were analysed in [2]. Three discrete families, each classified by an integer, of black hole solutions to the theory of a massless scalar field coupled repulsively to gravity¹ were found in [4]. More recently, it was shown in [5] that topologically massive gravity (TMG) [6] –Einstein gravity augmented by a gravitational Chern-Simons term– with a vanishing cosmological constant admits non-asymptotically flat (and non-asymptotically AdS), intrinsically non-static black hole solutions. The computation of the mass and angular momentum of these ACL black holes presented a challenge which was successively addressed and solved in [7].

The purpose of the present work is to construct and analyse black hole solutions to topologically massive gravitoelectrodynamics (TMGE), three-dimensional Einstein-Maxwell theory augmented by both gravitational and electromagnetic Chern-Simons terms. In [8], two classes of exact solutions of this theory were obtained by making suitable ansätze, the first leading to geodesically complete self-dual stationary solutions, and the other to diagonal solutions, including black point static solutions and anisotropic cosmologies. We will show in the following that a third, simple ansatz yields black hole solutions generalizing those of [5].

In the next section, we introduce the model, and summarize the dimensional reduction procedure followed in [8] to derive the field equations for solutions with two Killing vectors. Our ansatz leads in the third section to three black hole solutions (according to the values of the model parameters) depending generically on two integration constants. In the fourth section we analyse the global structure of our black hole spacetimes. After analytical extension through the two horizons, these are geodesically complete, but may allow closed timelike curves in certain parameter domains. We show that in the latter case it is possible to further narrow the parameter range so that the acausal region is hidden behind the event horizon. The mass, angular momentum and entropy of these black holes, computed in the fifth section, are checked to satisfy in all cases the first law of black hole thermodynamics for independent variations of the black hole parameters, as well as an integral Smarr-like relation. Finally, we show in the sixth section that our black hole metrics admit four local Killing vectors generating either the $sl(2, R) \times R$ algebra or (in a special case) a solvable Lie algebra. The existence of these four local isometries suggests that for a given set of model parameters the black hole solutions depending on different integration constants may be transformed into each other by local coordinate transformations, which we give

¹In contrast to the four-dimensional case, the gravitational constant can be either positive or negative in three dimensions [3].

explicitly. We close with a brief discussion.

2 The model

The action for TMGE may be written

$$I = I_E + I_M + I_{CSG} + I_{CSE}, \quad (2.1)$$

where

$$\begin{aligned} I_E &= \frac{1}{2\kappa} \int d^3x \sqrt{|g|} (R - 2\Lambda), \\ I_M &= -\frac{1}{4} \int d^3x \sqrt{|g|} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}. \end{aligned} \quad (2.2)$$

are the Einstein action (with cosmological constant Λ and Einstein gravitational constant $\kappa = 8\pi G$) and the Maxwell action, and

$$\begin{aligned} I_{CSG} &= \frac{1}{4\kappa\mu_G} \int d^3x \epsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^\rho \left[\partial_\mu \Gamma_{\rho\nu}^\sigma + \frac{2}{3} \Gamma_{\mu\tau}^\sigma \Gamma_{\nu\rho}^\tau \right], \\ I_{CSE} &= \frac{\mu_E}{2} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \end{aligned} \quad (2.3)$$

are the gravitational and electromagnetic Chern-Simons terms ($\epsilon^{\mu\nu\rho}$ is the antisymmetric symbol), with Chern-Simons coupling constants $1/\mu_G$ and μ_E .

We shall search for stationary circularly symmetric solutions of this theory using the dimensional reduction procedure of [8], which we summarize here. We choose the parametrisation [9, 10]

$$ds^2 = \lambda_{ab}(\rho) dx^a dx^b + \zeta^{-2}(\rho) R^{-2}(\rho) d\rho^2, \quad A = \psi_a(\rho) dx^a \quad (2.4)$$

($x^0 = t$, $x^1 = \varphi$), where λ is the 2×2 matrix

$$\lambda = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix}, \quad (2.5)$$

$R^2 \equiv \mathbf{X}^2$ is the Minkowski pseudo-norm of the “vector” $\mathbf{X}(\rho) = (T, X, Y)$,

$$\mathbf{X}^2 = \eta_{ij} X^i X^j = -T^2 + X^2 + Y^2, \quad (2.6)$$

and the scale factor $\zeta(\rho)$ allows for arbitrary reparametrizations of the radial coordinate ρ . We recall for future purpose that stationary solutions correspond to “spacelike” paths

$\mathbf{X}(\rho)$ with $R^2 > 0$, and that intersections of these paths with the future light cone ($R^2 = 0$, $T > 0$) correspond to event horizons.

The parametrization (2.4) reduces the action (2.1) to the form

$$I = \int d^2x \int d\rho L, \quad (2.7)$$

with the effective Lagrangian L

$$\begin{aligned} L = & \frac{1}{2} \left[\frac{1}{2\kappa\mu_G} \zeta^2 \mathbf{X} \cdot (\dot{\mathbf{X}} \wedge \ddot{\mathbf{X}}) + \frac{1}{2\kappa} \zeta \dot{\mathbf{X}}^2 \right. \\ & \left. + \zeta \dot{\bar{\psi}} \boldsymbol{\Sigma} \cdot \mathbf{X} \dot{\psi} + \mu_E \bar{\psi} \dot{\psi} - \frac{2}{\kappa} \zeta^{-1} \Lambda \right]. \end{aligned} \quad (2.8)$$

In (2.8), $\dot{} = \partial/\partial\rho$, the wedge product is defined by $(\mathbf{X} \wedge \mathbf{Y})^i = \eta^{ij} \epsilon_{jkl} X^k Y^l$ (with $\epsilon_{012} = +1$), the “Dirac” matrices Σ^i are

$$\Sigma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.9)$$

and $\bar{\psi} \equiv \psi^T \Sigma^0$ is the (real) Dirac adjoint of the “spinor” ψ .

Variation of the Lagrangian (2.8) with respect to ψ leads to the equation

$$\partial_\rho \left[\zeta (\boldsymbol{\Sigma} \cdot \mathbf{X}) \dot{\psi} + \mu_E \psi \right] = 0. \quad (2.10)$$

This means that the bracket is a constant of the motion, which may be set to zero by a gauge transformation, leading to the first integral

$$\zeta \dot{\psi} = \frac{\mu_E}{R^2} \boldsymbol{\Sigma} \cdot \mathbf{X} \psi. \quad (2.11)$$

This allows to eliminate from the Lagrangian (2.8) the spinor fields ψ and $\dot{\psi}$ in favor of the null ($\mathbf{S}_E^2 = 0$) “spin” vector field

$$\mathbf{S}_E = -\frac{\kappa}{2} \bar{\psi} \boldsymbol{\Sigma} \psi, \quad (2.12)$$

satisfying the equation (equivalent to (2.11))

$$\zeta \dot{\mathbf{S}}_E = \frac{2\mu_E}{R^2} \mathbf{X} \wedge \mathbf{S}_E. \quad (2.13)$$

Varying the Lagrangian (2.8) with respect to \mathbf{X} , we then obtain the dynamical equation for the vector fields \mathbf{X} ,

$$\begin{aligned} \ddot{\mathbf{X}} = & \frac{\zeta}{2\mu_G} \left[3(\dot{\mathbf{X}} \wedge \ddot{\mathbf{X}}) + 2(\mathbf{X} \wedge \dot{\ddot{\mathbf{X}}}) \right] \\ & - \frac{2\mu_E^2}{\zeta^2 R^2} \left[\mathbf{S}_E - \frac{2}{R^2} \mathbf{X} (\mathbf{S}_E \cdot \mathbf{X}) \right], \end{aligned} \quad (2.14)$$

where for simplicity we have fixed the scale $\zeta = \text{constant}$. Finally, variation of the Lagrangian (2.8) with respect to the Lagrange multiplier ζ leads to the Hamiltonian constraint

$$H \equiv \frac{1}{4\kappa} \left[\dot{\mathbf{X}}^2 + 2\mathbf{X} \cdot \ddot{\mathbf{X}} - \frac{\zeta}{\mu_G} \mathbf{X} \cdot (\dot{\mathbf{X}} \wedge \ddot{\mathbf{X}}) + 4\frac{\Lambda}{\zeta^2} \right] = 0, \quad (2.15)$$

where we have used the equation (following from (2.14))

$$\mathbf{S}_E \cdot \mathbf{X} = \frac{\zeta^2 R^2}{2\mu_E^2} \left[\mathbf{X} \cdot \ddot{\mathbf{X}} - \frac{3\zeta}{2\mu_G} \mathbf{X} \cdot (\dot{\mathbf{X}} \wedge \ddot{\mathbf{X}}) \right]. \quad (2.16)$$

In the preceding equations we have kept the constant scale parameter ζ free. From the circularly symmetric ansatz (3.1) we find $\det|g| = \zeta^{-2}$, showing that ζ has the same dimension (an inverse length) as the Chern-Simons coupling constants μ_G and μ_E . In the following it will prove convenient to fix the scale to

$$\zeta = \mu_E \quad (2.17)$$

(assumed, without loss of generality, to be positive), and to trade the gravitational Chern-Simons coupling constant μ_G for the dimensionless parameter

$$\lambda \equiv \frac{\mu_E}{2\mu_G}. \quad (2.18)$$

3 Black hole solutions

Two special classes of solutions to the reduced field equations (2.11)-(2.15) of TMGE have been given in [8]: self-dual solutions, which are asymptotically Minkowski or anti-de Sitter, and (in the case of exact balance $\mu_G + \mu_E = 0$) diagonal (static) solutions. In this paper, we shall derive the class of black hole solutions from the ansatz [13]

$$\mathbf{X} = \boldsymbol{\alpha} \rho^2 + \boldsymbol{\beta} \rho + \boldsymbol{\gamma}, \quad (3.1)$$

with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ linearly independent constant vectors.

We first insert this ansatz in the Hamiltonian constraint (2.15), which reduces to a second order equation in ρ . This is identically satisfied, provided the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are constrained by

$$\boldsymbol{\alpha}^2 = 0, \quad (\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) = 0, \quad (3.2)$$

$$\boldsymbol{\beta}^2 + 4 \left[\boldsymbol{\alpha} \cdot \boldsymbol{\gamma} + \lambda (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \cdot \boldsymbol{\gamma} + \frac{\Lambda}{\mu_E^2} \right] = 0. \quad (3.3)$$

It is easy to show that the two constraints (3.2) further imply

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = d\boldsymbol{\alpha}, \quad \boldsymbol{\beta}^2 = d^2, \quad (3.4)$$

for some constant d . Next, we compute from (2.16)

$$\mathbf{S}_E \cdot \mathbf{X} = b(\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})R^2, \quad b = 1 + 3d\lambda. \quad (3.5)$$

Inserting this into (2.14), we obtain

$$\mathbf{S}_E = b \left[2(\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})\mathbf{X} - \boldsymbol{\alpha}R^2 \right]. \quad (3.6)$$

Differentiating this, we obtain after some algebra

$$\begin{aligned} \dot{\mathbf{S}}_E &= 2b \left[-(\beta^2)\boldsymbol{\alpha}\rho + (\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})\boldsymbol{\beta} - (\boldsymbol{\beta} \cdot \boldsymbol{\gamma})\boldsymbol{\alpha} \right] \\ &= -2bd \left[d\boldsymbol{\alpha}\rho + \boldsymbol{\alpha} \wedge \boldsymbol{\gamma} \right], \end{aligned} \quad (3.7)$$

where the wedge product $\boldsymbol{\alpha} \wedge \boldsymbol{\gamma}$ has been evaluated with the aid of (3.4). On the other hand,

$$\mathbf{X} \wedge \mathbf{S}_E = b \left[d\boldsymbol{\alpha}\rho + \boldsymbol{\alpha} \wedge \boldsymbol{\gamma} \right] R^2. \quad (3.8)$$

Comparing Eqs. (3.7) and (3.8), we see that Eq. (2.13) for the gauge field is satisfied if either $b = 0$, leading to $d = -2\mu_G/3\mu_E$ (this is the case of TMG), or

$$d = -1. \quad (3.9)$$

In this case the remaining constraint equation (3.3) then leads to the value of the scalar product $\boldsymbol{\alpha} \cdot \boldsymbol{\gamma}$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\gamma} = -\frac{1 + 4\Lambda/\mu_E^2}{4(1 - \lambda)}. \quad (3.10)$$

At this stage the solution depends on five parameters (nine parameters in the ansatz (3.1) restricted by the four constraints $\boldsymbol{\alpha}^2 = 0$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = -\boldsymbol{\alpha}$ (two constraints), and (3.10)). In principle the number of free parameters may be reduced to two by taking into account the three-parameter group of transformations which leave the ansatz (3.1) (with $\zeta = \mu_E$ and the period of the angle φ fixed) form-invariant: translations of ρ , transition to uniformly rotating frames, and simultaneous length (ρ) and time (t) rescalings. However, for the sake of comparison with [14], we will allow for an arbitrary time scale parameter \sqrt{c} . Choose a rotating frame and a length-time scale such that $\boldsymbol{\alpha} = (c/2, -c/2, 0)$. In this frame,

$$\boldsymbol{\alpha} = (c/2, -c/2, 0), \quad \boldsymbol{\beta} = (\omega, -\omega, -1), \quad \boldsymbol{\gamma} = (z + u, z - u, v), \quad (3.11)$$

with $\boldsymbol{\alpha} \cdot \boldsymbol{\gamma} = -cz$ given by (3.10),

$$z = (1 - \beta^2)/2c, \quad (3.12)$$

where

$$\beta^2 = \frac{1 - 2\lambda - 4\Lambda/\mu_E^2}{2(1 - \lambda)}. \quad (3.13)$$

Computation of R^2 gives

$$R^2 = \beta^2 \rho^2 - 2(v + 2\omega z)\rho + v^2 - 4uz. \quad (3.14)$$

If $\beta^2 \neq 0$ the linear term can be set to zero ($v = -2\omega z$) by a translation of ρ , leading to

$$R^2 = \beta^2(\rho^2 - \rho_0^2), \quad (3.15)$$

where we have eliminated u in terms of z and the new real parameter ρ_0^2 . The final metric may be written in the two equivalent forms:

$$\begin{aligned} ds^2 = & \frac{1 - \beta^2}{c} \left[dt - \left(\frac{c\rho}{1 - \beta^2} + \omega \right) d\varphi \right]^2 - \frac{c\beta^2}{1 - \beta^2} (\rho^2 - \rho_0^2) d\varphi^2 \\ & + \frac{1}{\beta^2 \mu_E^2} \frac{d\rho^2}{\rho^2 - \rho_0^2}, \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} ds^2 = & -\beta^2 \frac{\rho^2 - \rho_0^2}{r^2} dt^2 + r^2 \left[d\varphi - \frac{\rho + (1 - \beta^2)\omega/c}{r^2} dt \right]^2 \\ & + \frac{1}{\beta^2 \mu_E^2} \frac{d\rho^2}{\rho^2 - \rho_0^2}, \end{aligned} \quad (3.17)$$

with

$$r^2 = c\rho^2 + 2\omega\rho + \frac{\omega^2}{c} (1 - \beta^2) + \frac{c\beta^2 \rho_0^2}{1 - \beta^2}. \quad (3.18)$$

This metric is very similar in form to that of [5]. While a superficial glance at (3.16) would suggest that the metric is Lorentzian provided $\beta^2 > 1$ for $c > 0$, or $\beta^2 < 1$ for $c < 0$, the ADM form (3.17) shows that the condition $\beta^2 > 0$ is sufficient. Obviously this is a black hole, with two horizons at $\rho = \pm\rho_0$, if $\rho_0^2 > 0$. The analysis of the causal structure of this spacetime, carried out in the next section, further shows that this is a causally regular black hole provided $c > 0$ and $\beta^2 < 1$. So the range of regularity of this black hole solution is

$$0 < \beta^2 < 1 \quad \Leftrightarrow \quad \begin{cases} 2\lambda > 1 - \frac{4\Lambda}{\mu_E^2} & \text{if } \Lambda < -\frac{\mu_E^2}{4}, \\ 2\lambda < 1 - \frac{4\Lambda}{\mu_E^2} & \text{if } \Lambda > -\frac{\mu_E^2}{4}. \end{cases} \quad (3.19)$$

The form of the metric breaks down for $\beta^2 = 1$ ($\Lambda = -\mu_E^2/4$) and $\beta^2 = 0$ ($2\lambda = 1 - 4\Lambda/\mu_E^2$), as well as in the intersection ($\lambda = 1$ with $\Lambda = -\mu_E^2/4$) of these two cases.

However it is possible to extend the solution to all these cases. We first consider the case $\beta^2 = 1$. In this case the parameter $z = 0$, and so also $v = 0$, and $\rho_0 = 0$, with the parameter u remaining free. So the metric in this case is (3.17) with

$$\beta^2 = 1, \quad \rho_0 = 0, \quad r^2 = c\rho^2 + 2\omega\rho + 2u. \quad (3.20)$$

In the case $\beta^2 = 0$ ($z = 1/2c$), the quadratic term is absent from (3.14). If $v + \omega/c \neq 0$ the constant term can be set to zero ($u = cv^2/2$) by a translation of ρ , so that (3.15) is replaced by

$$R^2 = 2\nu\rho, \quad (3.21)$$

with $\nu = -v - \omega/c > 0$. The metric may be written as

$$ds^2 = \frac{1}{c} \left[dt - c(\rho + \nu + \omega/c) d\varphi \right]^2 - 2c\nu\rho d\varphi^2 + \frac{d\rho^2}{2\mu_E^2\nu\rho}, \quad (3.22)$$

or

$$ds^2 = -\frac{2\nu\rho}{r^2} dt^2 + r^2 \left[d\varphi - \frac{\rho + \nu + \omega/c}{r^2} dt \right]^2 + \frac{d\rho^2}{2\mu_E^2\nu\rho}, \quad (3.23)$$

with

$$r^2 = c\rho^2 + 2\omega\rho + c(\nu + \omega/c)^2. \quad (3.24)$$

This is a causally regular black hole, with a single horizon at $\rho = 0$, if $c > 0$ and $\omega > -c\nu/2$. The formal limit $\nu \rightarrow 0$ corresponds to the exceptional case $v + \omega/c = 0$, with the horizonless metric (in which we have translated $\rho \rightarrow \rho - \omega/c$, and put $2u \equiv -c\rho_1^2$)

$$ds^2 = \frac{1}{c} \left[dt - c\rho d\varphi \right]^2 - c\rho_1^2 d\varphi^2 + \frac{d\rho^2}{\mu_E^2\rho_1^2} \quad (3.25)$$

$$= -\frac{\rho_1^2}{c(\rho^2 - \rho_1^2)} dt^2 + c(\rho^2 - \rho_1^2) \left[d\varphi - \frac{\rho}{c(\rho^2 - \rho_1^2)} dt \right]^2 + \frac{d\rho^2}{\mu_E^2\rho_1^2}. \quad (3.26)$$

Finally, in the case $\lambda \equiv 1$ ($\mu_E/\mu_G = 2$) with $\Lambda = -\mu_E^2/4$, the constraint (3.3) is *identically* satisfied for all values of the scalar product $\alpha \cdot \gamma = -z$, i.e. for all values of β^2 . So the regular black hole solution is again given by the generic forms (3.16) or (3.17) for $0 < \beta^2 \leq 1$ (with $\rho_0 = 0$ for $\beta^2 = 1$), where now β^2 is a *free* parameter, or by (3.22) or (3.23) in the limiting case $\beta^2 = 0$.

The electromagnetic field generating this gravitational field may be obtained by solving Eq. (2.12) for ψ , with \mathbf{S}_E given by (3.6). This gives

$$A = \pm \sqrt{-\frac{c(1-3\lambda)}{\kappa}} \left[\frac{(1-\beta^2)}{c} dt - (\rho + (1-\beta^2)\omega/c) d\varphi \right] \quad (3.27)$$

for $0 < \beta^2 \leq 1$, or

$$A = \pm \sqrt{-\frac{c(1-3\lambda)}{\kappa}} \left[c^{-1} dt - (\rho + \nu + \omega/c) d\varphi \right] \quad (3.28)$$

for $\beta^2 = 0$. Clearly this field is purely magnetic, as the constant electric potential can be set to zero by a gauge transformation.

The reality of this electromagnetic potential leads to an additional restriction on the domain of existence of these black hole solutions. Recall that in (2+1)-dimensional Einstein gravity, the sign of the gravitational constant κ is not fixed *a priori* [3]. It has been argued [6] that in the case of topologically massive gravity the gravitational constant should be taken negative to avoid the appearance of ghosts. This argument follows from considering the TMG action linearized around Minkowski spacetime. However, similarly to the case of the BTZ black hole solutions of (2+1)-dimensional gravity with a negative cosmological constant [1], it might be more appropriate to linearize the TMGE action around a suitable “vacuum” solution belonging to the black hole family, and it is not clear in this case what sign of κ should be taken. We shall consider both signs to be possible. For $\kappa c > 0$, the ratio of the two Chern-Simons coupling constants must be bounded by $\mu_E/\mu_G > 2/3$. This leads to regular black holes ($0 \leq \beta^2 \leq 1$) only if $c > 0$ and $\Lambda < \mu_E^2/12$. For $\kappa c < 0$, the bound is inversed, $\mu_E/\mu_G < 2/3$. This leads to regular black holes if $c > 0$ and $\Lambda \geq -\mu_E^2/4$.

Finally the electromagnetic field vanishes for $\mu_E/\mu_G = 2/3$. So, when the two Chern-Simons coupling constants are constrained by this relation, the black hole metrics (3.16) or (3.17) (or (3.22) or (3.23) for $\Lambda = \mu_E^2/12 = \mu_G^2/27$) again solve the equations of TMG. For $c < 0$, these $\lambda = 1/3$ solutions coincide (after appropriate coordinate transformations) with the black hole solutions to TMG with cosmological constant given in [11, 12] (see also [13], Eq. (18)). Regular black holes (with $c > 0$) were not correctly identified in these papers. For $\Lambda = 0$ ($\beta^2 = 1/4$) and $c = 1/4$, the black hole metrics (3.16) and (3.17) reduce respectively, after rescaling $\rho \rightarrow 2\rho$, $\rho_0 \rightarrow 2\rho_0$ and in units such that $\mu_G = 3$, to Eqs. (4) and (6) of [5].

The electromagnetic field also vanishes in the exceptional case $c = 0$, corresponding to $\alpha = 0$. In this case the metric [9] $\mathbf{X} = \beta\rho + \gamma$, with $\beta^2 = -4\Lambda/\mu_E^2$ from (3.3), describes for $\Lambda < 0$ the well-known BTZ black holes [1].

To close this section we comment on the relation between our black hole solutions and those of [14]. In the limit $\lambda \rightarrow 0$ ($\mu_G \rightarrow \infty$), the gravitational Chern-Simons coupling constant goes to zero, and TMGE reduces for a negative cosmological constant $\Lambda = -\ell^{-2}$ to the Einstein-Maxwell-Chern-Simons theory considered in [14], with $\alpha = -\mu_E/2$. For $\kappa = 8\pi G$ positive (as assumed in [14]), the electromagnetic field (3.27) can be real only if $c < 0$, so that the solutions necessarily admit naked closed timelike curves. Noting that

our constant β^2 is given for $\mu_G \rightarrow \infty$ by

$$1 - \beta^2 = \frac{\alpha^2 \ell^2 - 1}{2\alpha^2 \ell^2}, \quad (3.29)$$

we find that the ‘‘Gödel cosmon’’ solution ((17) of [14]) with $\alpha^2 \ell^2 > 1$ and the ‘‘Gödel black hole’’ solution ((31) of [14]) can both be put in the form (3.16) or (3.17), where our radial coordinate ρ is related to the radial coordinate \bar{r} of [14] by $\rho = 2\alpha\bar{r} - (1 - \beta^2)\omega/c$, with $c = \mp(1 - \beta^2)$ (upper sign in the case of (17) and lower sign in the case of (31)), and our integration constants ω and ρ_0 are related to the constants ν and J by

$$\omega = \Gamma\nu, \quad \rho_0^2 = \Gamma^2\nu^2 \pm 2\Gamma J \quad \left(\Gamma = \frac{2G}{\alpha\beta^2} = \frac{4\alpha\ell^2 G}{1 + \alpha^2 \ell^2} \right). \quad (3.30)$$

4 Global structure

Although the black hole spacetimes obtained in the previous section are not constant curvature, their curvature invariants are constant,

$$\begin{aligned} \mathcal{R} &= \frac{1 - 4\beta^2}{2} \mu_E^2, \\ \mathcal{R}_{\mu\nu} R^{\mu\nu} &= \frac{3 - 8\beta^2 + 8\beta^4}{4} \mu_E^4, \end{aligned} \quad (4.1)$$

and depend only on the parameter β^2 . These spacetimes are clearly regular for all $\rho \neq \pm\rho_0$, and may (except in the special case $\omega = \rho_0/\sqrt{1 - \beta^2}$, see below) be extended through the horizons $\rho = \pm\rho_0$ by the usual Kruskal method, leading to geodesically complete spacetimes. However they may have closed timelike curves (CTC) for certain ranges of their parameters. The circles $\rho = \text{constant}$ are timelike whenever $g_{\varphi\varphi} = r^2 < 0$. For $c < 0$, r^2 is negative at spacelike infinity ($\rho \rightarrow \infty$), so these solutions always admit CTC outside the outer horizon. To exclude these we will assume in the following $c > 0$. Fixing the scale so that $c = +1$, we find that in the case of the generic black hole (3.17), the zeroes of r^2 are located at

$$\rho_{\pm} = -\omega \pm \beta \sqrt{\omega^2 - \frac{\rho_0^2}{1 - \beta^2}}. \quad (4.2)$$

It follows that CTC are absent for $\beta^2 < 1$ and

$$\omega^2 < \rho_0^2 / (1 - \beta^2). \quad (4.3)$$

The Penrose diagram for the maximally extended spacetime is then similar to that of the Kerr black hole. For $\beta^2 < 1$ and $\omega^2 > \rho_0^2 / (1 - \beta^2)$, CTC do occur in the range $\rho \in [\rho_-, \rho_+]$,

but are hidden behind the two horizons ($\rho_- < \rho_+ < -\rho_0$) if $\omega > 0$. The Penrose diagram with the acausal regions cut out is the same as for Reissner-Nordström black holes. For other values of β or ω , CTC occur outside the outer horizon. The limiting case $\rho_0 = 0$ corresponds to extreme black holes, with an acausal region behind the horizon for $\omega > 0$. The corresponding Penrose diagram (again with the acausal regions cut out) is similar to that of extreme Reissner-Nordström black holes. In the exceptional case $\omega = \rho_0/(1 - \beta^2)$, the inner horizon $-\rho_0$ coincides with the outer boundary ρ_+ of the acausal region, with the metric reducing to

$$ds^2 = -\beta^2 \frac{\rho - \rho_0}{\rho + (1 + \beta^2)\omega} dt^2 + \left(\rho + (1 + \beta^2)\omega \right) (\rho + \rho_0) \left(d\varphi - \frac{dt}{\rho + (1 + \beta^2)\omega} \right)^2 + \frac{1}{\beta^2 \mu_E^2} \frac{d\rho^2}{\rho^2 - \rho_0^2}. \quad (4.4)$$

This has only one horizon at $\rho = +\rho_0$, where Kruskal extension can be carried out as usual. As discussed in [5] (for $\beta^2 = 1/4$), geodesics actually terminate at the causal singularity $\rho = -\rho_0$, which is thus a true spacelike singularity of the metric (4.4). The resulting Penrose diagram is similar to that of the Schwarzschild black hole.

Similarly, in the case $\beta^2 = 1$ ($\rho_0 = 0$), with r^2 given by (3.20), we find

$$\rho_{\pm} = -\omega \pm \sqrt{\omega^2 - 2u}, \quad (4.5)$$

so that CTCs are absent if $\omega^2 < 2u$. The horizon being double, the corresponding Penrose diagram is similar to that of the extreme Kerr black hole. If $u > 0$, CTC do exist but are hidden behind the double horizon if $\omega > \sqrt{2u}$. The Penrose diagram in this case is identical to that of the extreme Reissner-Nordström black hole. In the exceptional case $u = 0$, the metric reduces to

$$ds^2 = -\frac{\rho}{\rho + 2\omega} dt^2 + \rho(\rho + 2\omega) \left(d\varphi - \frac{dt}{\rho + 2\omega} \right)^2 + \frac{d\rho^2}{\mu_E^2 \rho^2} \quad (4.6)$$

(the $\beta^2 = 1$, $\rho_0 = 0$ limit of (4.4)), showing that the would-be horizon $\rho = 0$ is actually a null singularity.

Finally, in the case $\beta^2 = 0$ (solution(3.23)-(3.24)),

$$c\rho_{\pm} = -\omega \pm \sqrt{\omega^2 - (\nu + \omega)^2} \quad (4.7)$$

($\nu > 0$), so that if $\omega > -\nu/2$ CTCs are absent. There is a single horizon at $\rho = 0$, and the reduced two-dimensional (t, ρ) metric patches $\rho > 0$ and $\rho < 0$ are both asymptotically conformal to Minkowski 2-space. Accordingly, the Penrose diagram of this geodesically complete spacetime is similar to that of the Rindler metric. On the other hand, if $\omega \leq -\nu/2$, $\rho_- > 0$, leading to naked CTCs.

As in the case of the rotating black holes of TMG [5], the price to pay for the causal regularity of our black holes is that the Killing vector ∂_t is not timelike, but spacelike (or null for $\beta^2 = 1$), so that no observer can remain static outside the black hole. In other words, the TMGE black holes are surrounded by an ergosphere extending from the outer horizon to infinity. However locally stationary observers are allowed. Their worldline must remain timelike ($ds^2 < 0$), so that for $\rho \geq \rho_0$ fixed, the angular velocity $\Omega \equiv d\varphi/dt$ is constrained by the inequality:

$$r^2\Omega^2 - 2\Omega(\rho + \omega(1 - \beta^2)) + 1 - \beta^2 < 0. \quad (4.8)$$

This is satisfied by

$$\frac{\rho + \omega(1 - \beta^2) - \beta\sqrt{\rho^2 - \rho_0^2}}{r^2} < \Omega < \frac{\rho + \omega(1 - \beta^2) + \beta\sqrt{\rho^2 - \rho_0^2}}{r^2}, \quad (4.9)$$

which simplifies in the “vacuum” case $\omega = \rho_0 = 0$ to

$$1 - \beta < \Omega\rho < 1 + \beta. \quad (4.10)$$

In the limiting case $\beta^2 = 1$ with $\rho_0 = 0$, the local stationarity condition (4.9) becomes

$$0 < \Omega < \frac{2\rho}{r^2}. \quad (4.11)$$

Finally, in the special case $\beta^2 = 0$, the local stationarity condition is replaced by

$$\frac{\rho + \omega + \nu - \sqrt{2\nu\rho}}{r^2} < \Omega < \frac{\rho + \omega + \nu + \sqrt{2\nu\rho}}{r^2} \quad (4.12)$$

for the generic black solution (3.23), or

$$\frac{1}{\rho + \rho_1} < \Omega < \frac{1}{\rho - \rho_1} \quad (4.13)$$

for the vacuum solution (3.26).

5 Mass, angular momentum and entropy

The mass and angular momentum of black hole solutions of TMGE linearized about an appropriate background are the Killing charges, defined as integrals over the boundary ∂M of a spacelike hypersurface M

$$Q(\xi) = \frac{1}{\kappa} \int_{\partial M} \sqrt{|g|} \mathcal{F}^{0i}(\xi) dS_i, \quad (5.1)$$

of the superpotentials $\mathcal{F}^{\mu\nu}$ associated with the Killing vectors ∂_t and ∂_φ . These superpotentials may be written as the sum

$$\mathcal{F}^{\mu\nu}(\xi) = \mathcal{F}_g^{\mu\nu}(\xi) + \mathcal{F}_e^{\mu\nu}(\xi), \quad (5.2)$$

where the purely gravitational contribution ($A_\mu = 0$) is given in [7] (the sum of Eqs. (2.14) and (2.21)), and the electromagnetic contribution is given in [14] (the sum of Eqs. (68) and (71)). Rescaling the electromagnetic fields of [14] by a factor 2κ , this electromagnetic contribution is

$$\begin{aligned} \mathcal{F}_e^{\mu\nu}(\xi) = & \frac{\kappa}{\sqrt{|\hat{g}|}} \delta \left[\sqrt{|g|} F^{\mu\nu} - \mu_E \epsilon^{\mu\nu\lambda} A_\lambda \right] \xi^\rho \hat{A}_\rho \\ & + \kappa \left[\hat{F}^{\mu\nu} \xi^\rho + \hat{F}^{\nu\rho} \xi^\mu + \hat{F}^{\rho\mu} \xi^\nu \right] \delta A_\rho, \end{aligned} \quad (5.3)$$

where the hatted fields are those of the background (or “vacuum”), and δ stands for the difference between the fields evaluated for the black hole configuration and for the background configuration. Let us evaluate the radial component $\mathcal{F}_e^{02}(\xi)$ in the adapted coordinates of (2.4) and the adapted gauge of (2.11). We recognize in the first bracket of (5.3) the constant of the motion which was set to zero by the gauge fixing (2.11), and there remains

$$\mathcal{F}_e^{02}(\xi) = -\kappa \hat{F}^{2a} \epsilon_{ab} \xi^b \delta \psi_1 = -\kappa \zeta^2 (\xi^T \hat{\psi}) \delta \bar{\psi}^0, \quad (5.4)$$

which may be rearranged as

$$\mathcal{F}_e^{02}(\xi) = \frac{\zeta^2}{2} \left[\xi^T \boldsymbol{\Sigma} \cdot \delta \mathbf{S}_E - \kappa (\delta \bar{\psi} \hat{\psi}) \xi^T \right]^0. \quad (5.5)$$

Combining this with the gravitational contribution given in [7]², we obtain the net Killing charge for TMGE,

$$Q(\xi) = \frac{\pi \zeta}{\kappa} \left\{ \xi^T \left(\boldsymbol{\Sigma} \cdot \delta \mathbf{J} + \Delta \right) \right\}^0, \quad (5.6)$$

where \mathbf{J} is the constant super angular momentum [8]

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_G + \mathbf{S}_E, \quad \mathbf{L} = \mathbf{X} \wedge \dot{\mathbf{X}}, \quad \mathbf{S}_G = \lambda \left[\dot{\mathbf{X}} \wedge \mathbf{L} - 2\mathbf{X} \wedge \dot{\mathbf{L}} \right], \quad (5.7)$$

and Δ is the scalar

$$\Delta = \hat{\mathbf{X}} \cdot \delta \dot{\mathbf{X}} + \lambda [\dot{\hat{\mathbf{X}}} \cdot \delta \mathbf{L} - 2\hat{\mathbf{X}} \cdot \delta \dot{\mathbf{L}}] - \kappa (\delta \bar{\psi} \hat{\psi}). \quad (5.8)$$

²The matrices $\boldsymbol{\tau}$ of [7] are equal to $-\boldsymbol{\Sigma}^T$.

The mass and angular momentum are respectively the Killing charges for the vectors $\xi = (-1, 0)$ and $\xi = (0, 1)$:

$$M = -\frac{\pi\zeta}{\kappa}(\delta J^Y + \Delta), \quad J = \frac{\pi\zeta}{\kappa}(\delta J^T - \delta J^X). \quad (5.9)$$

We shall check that the values of these observables are consistent with the first law of black hole thermodynamics,

$$dM = T_H dS + \Omega_h dJ, \quad (5.10)$$

where the other observables, readily computed from the metric in ADM form

$$ds^2 = -N^2 dt^2 + r^2(d\varphi + N^\varphi dt)^2 + \frac{1}{(\zeta r N)^2} d\rho^2, \quad (5.11)$$

are the Hawking temperature and the horizon angular velocity

$$T_H = \frac{1}{4\pi}\zeta r_h(N^2)'(\rho_0), \quad \Omega_h = -N^\varphi(\rho_0) \quad (5.12)$$

(with $r_h = r(\rho_0)$ the horizon areal radius), and the black hole entropy S , which is the sum of the familiar Einstein contribution and a gravitational Chern-Simons contribution [15, 16, 7]

$$S = \frac{4\pi^2}{\kappa} \left(r_h - \lambda r_h^3 (N^\varphi)'(\rho_0) \right). \quad (5.13)$$

Let us discuss separately the various cases:

a) In the generic case $0 < \beta^2 < 1$ (solution (3.17)), the natural vacuum is the horizon-less metric

$$ds^2 = -\beta^2 dt^2 + \frac{1}{\beta^2 \mu_E^2} \frac{d\rho^2}{\rho^2} + \rho^2 \left[d\varphi - \frac{dt}{\rho} \right]^2 \quad (5.14)$$

(the $\omega = 0$ member of the extreme black hole $\rho_0 = 0$ family). The corresponding observables³

$$\begin{aligned} M &= \frac{2\pi\mu_E}{\kappa}(1-\lambda)\beta^2(1-\beta^2)\omega, \\ J &= \frac{\pi\mu_E}{\kappa}\beta^2 \left[(1-\lambda)(1-\beta^2)\omega^2 - \frac{1-\lambda(1-2\beta^2)}{1-\beta^2}\rho_0^2 \right], \\ S &= \frac{4\pi^2}{\kappa\sqrt{1-\beta^2}} \left[(1-\lambda)(1-\beta^2)\omega + (1-\lambda(1-2\beta^2))\rho_0 \right], \\ \Omega_h &= \frac{1-\beta^2}{\rho_0 + \omega(1-\beta^2)}, \quad T_H = \frac{\mu_E}{2\pi} \frac{\beta^2 \sqrt{1-\beta^2} \rho_0}{\rho_0 + \omega(1-\beta^2)} \end{aligned} \quad (5.15)$$

³In all cases the two terms δJ^Y and Δ of the r.h.s. of the first equation (5.9) contribute equally to the mass M , which is therefore twice the super angular momentum value $-(\pi\zeta/\kappa)\delta J^Y$ as computed in [5].

satisfy the first law (5.10) for independent variations of the black hole parameters ω and ρ_0 . Let us discuss under what conditions the mass M and the entropy S are positive:

(α) $1/3 \leq \lambda < 1$ ($\kappa > 0$). The mass is positive for $\omega > 0$. The entropy is then positive definite.

(β) $\lambda > 1$ ($\kappa > 0$). The mass is positive for $\omega < 0$, while the entropy is positive if

$$\omega < \frac{1 - \lambda(1 - 2\beta^2)}{(\lambda - 1)(1 - \beta^2)}\rho_0. \quad (5.16)$$

This upper bound for ω is consistent with the lower bound ensuring causal regularity, $\omega > -\rho_0/\sqrt{1 - \beta^2}$.

(γ) $-1/3 \leq \lambda \leq 1/3$ ($\kappa < 0$). The conditions for the positivity of the mass and entropy are the same as in case (β), however the upper bound (5.16) is not consistent with causal regularity.

(δ) $\lambda < -1/3$ ($\kappa < 0$). The positivity conditions are again the same as in case (β), the upper bound (5.16) being conditionally consistent with causal regularity if $\lambda > -1$, and always consistent if $\lambda \leq -1$.

b) In the case $\beta^2 = 1$ ($\rho_0 = 0$), the black hole parameters are ω and $2u$ ($2u > 0$). The vacuum metric (5.14) corresponds to the parameter values $\omega = 2u = 0$. As can be expected from the limit of the generic case (5.15), the values of the observables considerably simplify with the mass, horizon angular velocity and Hawking temperature vanishing, and the angular momentum and entropy independent of ω ,

$$\begin{aligned} M &= 0, & J &= -\frac{\pi\mu_E}{\kappa}(1 + \lambda)2u, & S &= \frac{4\pi^2}{\kappa}(1 + \lambda)\sqrt{2u}, \\ \Omega_h &= 0, & T_H &= 0. \end{aligned} \quad (5.17)$$

The first law is trivially satisfied. The entropy is positive either for $\lambda \geq 1/3$ ($\kappa > 0$) or for $\lambda < -1$ ($\kappa < 0$).

c) In the case $\beta^2 = 0$, the black hole parameters are ω and ν ($\nu > 0$) with the vacuum metric (3.26) corresponding to the limit $\nu \rightarrow 0$. The observables are

$$\begin{aligned} M &= -\frac{2\pi\mu_E}{\kappa}(1 - \lambda)\nu, & J &= -\frac{2\pi\mu_E}{\kappa}\nu \left[(1 - \lambda)\omega + \nu \right], \\ S &= \frac{4\pi^2}{\kappa} \left[(1 - \lambda)\omega + (1 + \lambda)\nu \right], \\ \Omega_h &= \frac{1}{\nu + \omega}, & T_H &= \frac{\mu_E}{2\pi} \frac{\nu}{\nu + \omega}. \end{aligned} \quad (5.18)$$

The first law is satisfied for independent variations of the black hole parameters. The discussion of the positivity of mass and entropy parallels that made in the generic case,

with some modifications: in the case (α) ($1/3 \leq \lambda < 1$) the mass is now negative definite, and in the other cases the upper bound (5.16) should be replaced by

$$\omega < \frac{1+\lambda}{\lambda-1}\nu. \quad (5.19)$$

d) Finally in the exceptional case $\lambda = 1$ ($\kappa > 0$), the metric generically depends on the three free parameters β^2 , ρ_0 and ω . However the mass vanishes, so that according to the first law there should be only one free parameter to vary. Indeed, for all values of β^2 , we can take this parameter to be the entropy S , which is positive definite. The other observables are related to this by

$$J = -\frac{\kappa\mu_E}{32\pi^3}S^2, \quad T_H = \frac{\kappa\mu_E}{16\pi^3}\Omega_h S, \quad (5.20)$$

and the first law is clearly satisfied.

To conclude this section we note that, in all the preceding cases, the TMGE black hole observables satisfy, besides the differential first law (5.10), the integral Smarr-like relation

$$M = T_H S + 2\Omega_h J. \quad (5.21)$$

This is the natural generalisation of the Smarr-like relation given in [17], Eq. (2.30), where in the case of TMGE M should be replaced by $M/2$ in accordance with the remark made in footnote 2.

6 Symmetries

6.1 Killing vectors

The ACL black holes were constructed in [5] by analytic extension of the Vuorio solution, from which they inherited the local isometry algebra $\text{Lie}[SL(2, R) \times U(1)]$ [18, 19]. This property generalizes to the case of TMGE black holes with $\beta^2 > 0$. The metric (3.17) admits four local Killing vectors⁴ $L_t = \partial_t$, and L_0 , $L_{\pm 1}$ generating the $sl(2, R)$ algebra

$$[L_m, L_n] = (m - n)L_{m+n} \quad (m, n = -1, 0, 1), \quad (6.1)$$

with $[L_t, L_n] = 0$. The L_n are, in the case of generic $\beta^2 > 0$ black holes,

$$\begin{aligned} L_0 &= \alpha^{-1}(\partial_\varphi + \omega\partial_t), \\ L_{\pm 1} &= e^{\mp\alpha\varphi} \left[\frac{1}{\alpha\sqrt{\rho^2 - \rho_0^2}} \left(\rho(\partial_\varphi + \omega\partial_t) + \frac{\alpha^2}{\beta^4\mu_E^2}\partial_t \right) \pm \sqrt{\rho^2 - \rho_0^2}\partial_\rho \right], \end{aligned} \quad (6.2)$$

⁴As usual the periodicity condition on φ allows only the two global Killing vectors ∂_t and ∂_φ .

with

$$\begin{aligned}\alpha &= \frac{\rho_0}{\gamma} \left(\gamma = \frac{\sqrt{1-\beta^2}}{\beta^2 \mu_E} \right) & \text{for } \beta^2 < 1, (\rho_0 \neq 0), \\ \alpha &= \mu_E \sqrt{2u} & \text{for } \beta^2 = 1, (u \neq 0),\end{aligned}\tag{6.3}$$

or, in the case of extreme $\beta^2 > 0$ black holes ($\beta^2 < 1$ with $\rho_0 = 0$ or $\beta^2 = 1$ with $u = 0$),

$$\begin{aligned}L_1 &= \partial_\varphi + \omega \partial_t, \\ L_0 &= \varphi(\partial_\varphi + \omega \partial_t) - \rho \partial_\rho, \\ L_{-1} &= \left(\varphi^2 + \frac{\gamma^2}{\rho^2} \right) (\partial_\varphi + \omega \partial_t) - 2\varphi \rho \partial_\rho + \frac{2}{\beta^4 \mu_E^2} \rho^{-1} \partial_t.\end{aligned}\tag{6.4}$$

In the exceptional case $\beta^2 = 0$, the algebra of local isometries is instead the solvable Lie algebra

$$\begin{aligned}[L_{\pm 1}, L_0] &= \pm L_{\pm 1}, \\ [L_1, L_{-1}] &= -2L_t,\end{aligned}\tag{6.5}$$

$$[L_t, L_n] = 0\tag{6.6}$$

($n = -1, 0, 1$), for the four local Killing vectors $L_t = \partial_t$ and, in the case of generic black holes,

$$\begin{aligned}L_0 &= \alpha^{-1}(\partial_\varphi + \omega \partial_t), \\ L_{\pm 1} &= e^{\mp \alpha \varphi} \left[\frac{1}{\sqrt{2\alpha\rho}} \left(\partial_\varphi + \omega \partial_t - (\rho - \nu) \partial_t \right) \pm \sqrt{2\alpha\rho} \partial_\rho \right],\end{aligned}\tag{6.7}$$

with $\alpha = \mu_E \nu \neq 0$, or, for the vacuum $\nu = 0$,

$$\begin{aligned}L_0 &= \gamma^{-2} \rho (\partial_\varphi + \omega \partial_t) + \gamma^2 \varphi \partial_\rho + \frac{1}{2} \left(\gamma^{-2} \rho^2 + \gamma^2 \varphi^2 \right) \partial_t, \\ L_{\pm 1} &= \gamma^{-1} (\partial_\varphi + \omega \partial_t) \pm \gamma (\partial_\rho + \varphi \partial_t),\end{aligned}\tag{6.8}$$

with $\gamma^2 = \mu_E \rho_1^2$.

6.2 Generating black holes from vacuum

The fact that the curvature invariants (4.1) depend only on β^2 , and that for a given value of β^2 the local Killing vectors for our black holes yield different realizations of the same Lie algebra suggests that these solutions may be transformed into each other by local coordinate transformations. Again we must treat separately the case of the generic solution (3.16) and the two special cases $\beta^2 = 0$ and $\beta^2 = 1$.

a) The metric (3.16) may be reduced *à la* Kaluza-Klein to an AdS_2 (φ, ρ) metric. This is similar to the case of the four-dimensional metric RBR^- considered in [20], and may be generated from the vacuum metric in the same manner. Writing (3.16) as

$$ds^2 = -\frac{4}{\beta^2 \mu_E^2} \frac{(\rho^2 - \rho_0^2)}{\rho_0^2} du_+ du_- + (1 - \beta^2) \left(dt - \omega d\varphi - \frac{\rho}{1 - \beta^2} d\varphi \right)^2, \quad (6.9)$$

with

$$u_{\pm} = \frac{1}{2} \left(\frac{\rho_0}{\gamma} \varphi \pm \xi \right), \quad (6.10)$$

where γ is defined in (6.3), and the coordinate ξ is related to ρ by

$$\rho = \rho_0 \coth \xi, \quad (6.11)$$

we see that it can be obtained from the vacuum ($\omega = \rho_0 = 0$) metric (hatted coordinates) by the combined transformation

$$\begin{aligned} \hat{u}_{\pm} &\equiv \frac{1}{2} \left(\frac{\rho_0}{\gamma} \hat{\varphi} \pm \frac{\rho_0}{\hat{\rho}} \right) = \tanh u_{\pm}, \\ \hat{t} &= t - \omega \varphi + \frac{\gamma}{1 - \beta^2} \ln \left(\frac{\cosh u_-}{\cosh u_+} \right). \end{aligned} \quad (6.12)$$

b) In the case $\beta^2 = 1$, the transformation between the black hole metric

$$ds^2 = \frac{1}{\beta^2 \mu_E^2} \frac{d\rho^2}{\rho^2} + (\rho^2 + 2u) d\varphi^2 - 2\rho d\varphi (dt - \omega d\varphi) \quad (6.13)$$

and the vacuum metric $\omega = 2u = 0$ is less obvious. However it may be easily be obtained from the preceding case by putting $\gamma = \rho_0/\alpha$ and taking the limit $\rho_0 \rightarrow 0$ with $\alpha = \mu_E \sqrt{2u}$ fixed, leading to

$$\begin{aligned} \hat{\rho} &= \rho \cosh^2 \left(\frac{\alpha}{2} \varphi \right), \\ \hat{\varphi} &= \frac{2}{\alpha} \tanh \left(\frac{\alpha}{2} \varphi \right), \\ \hat{t} &= t - \omega \varphi - \frac{\alpha}{\mu_E^2 \rho} \tanh \left(\frac{\alpha}{2} \varphi \right). \end{aligned} \quad (6.14)$$

c) The case $\beta^2 = 0$ (metric (3.22)) is similar to the generic case, except that AdS_2 is replaced by the flat Rindler spacetime. Putting $\rho = (\mu_E^2 \nu / 2) x^2$, (3.22) may be written

$$ds^2 = dx^2 - \mu_E^2 \nu^2 x^2 d\varphi^2 + \left(dt - (\omega + \nu) d\varphi - \frac{\mu_E^2 \nu}{2} x^2 d\varphi \right)^2. \quad (6.15)$$

This can be obtained from the vacuum metric (3.25) (hatted coordinates) by the combined transformation

$$\begin{aligned}\hat{\rho} &= \mu_E \rho_1 x \cosh(\mu_E \nu \varphi), \\ \hat{\varphi} &= \frac{x}{\rho_1} \sinh(\mu_E \nu \varphi), \\ \hat{t} &= t - (\omega + \nu)\varphi + \frac{\mu_E x^2}{4} \sinh(2\mu_E \nu \varphi).\end{aligned}\tag{6.16}$$

7 Discussion

We have constructed intrinsically rotating black hole solutions to three-dimensional Einstein-Maxwell theory with both gravitational and electromagnetic Chern-Simons terms. These are geodesically complete and causally regular within a certain parameter range. We have computed the mass, angular momentum and entropy of these black holes, and checked that they satisfy the first law of black hole thermodynamics. We have also shown that these Chern-Simons black holes admit a four-parameter local isometry algebra, which generically is $sl(2, R) \times R$, and that they may be generated from the corresponding vacua by local coordinate transformations.

As the write-up of this paper was being finalized, we learned about the paper [21], which partly overlaps the present work. This is concerned with pure TMG, so that our parameter β^2 is related to their $\nu^2 = -\mu_G^2/9\Lambda$ by $\beta^2 = (\nu^2 + 3)/4\nu^2$. Their ‘spacelike stretched black holes’ with $\nu^2 > 1$ correspond to our generic $\beta^2 < 1$ black holes, while their null warped black hole (6.14) with $\nu^2 = 1$ corresponds to our $\beta^2 = 1$ black hole. They also exhibit local coordinate transformations generating their $\nu^2 > 1$ black holes from their spacelike warped AdS_3 (3.3) (which can be obtained from our solution (3.16) by setting e.g. $c = 1 - \beta^2$, $\rho_0^2 = -1$, $\omega = 0$, and making the coordinate transformation $\rho = \sinh \sigma$, $t = \gamma u$, $\varphi = -\gamma \tau$, with $\gamma = 2\nu\ell/(\nu^2 + 3)$), and their $\nu^2 = 1$ black hole from their null warped AdS_3 .

This work should be extended in several directions. The close analogy between our generic black hole metric (3.16) and the rotating Bertotti-Robinson metric RBR_- of [20] strongly suggests that the geodesic and test scalar field equations can similarly be separated and solved in the present case. Likewise, an investigation of the asymptotic symmetries of our black holes should extend the local $sl(2, R) \times R$ isometry algebra to a Virasoro algebra. In another vein, our ansatz (3.1) can be extended to yield black hole solutions to three-dimensional Einstein-Maxwell-dilaton theory with an electromagnetic Chern-Simons term. These shall be reported and discussed elsewhere [22].

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